Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021–8928(03)00092-3
J. Appl. Maths Mechs, Vol. 67, No. 4, pp. 539–552, 2003

0021–8928/$—see front matter

A ROBOT-BALANCER ON A CYLINDER†

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(*Received* 12 *March* 2003)

The problem of stabilizing the equilibrium of a robot placed on a cylinder which can roll along a horizontal plane is investigated. There is no slip in any of the external contacts. Control is achieved by means of the electromechanical angular acceleration of a flywheel on the robot. Steady motions are studied. The basic procedures for stabilizing the robot in a vertical position are analysed in a non-linear formulation. It is shown that the corresponding linear system is completely controllable. A coordinate and velocity controller with saturation is constructed. The domain in which the system can be stabilized is found in connection with the boundedness of the control function. The effect of measurement errors is examined. The control characteristics are calculated for certain actual robot parameters.

The problem of the active stabilization of mechanical systems in the neighbourhood of an unstable equilibrium position has numerous applications. Among these, we mention the problem of ensuring the equilibrium of an inverted pendulum. The need for its solution arises, in particular, when designing stable walking and running walking machines with a small number of feet.

There are various methods of ensuring the stability of the equilibrium of an inverted pendulum, for example, by a suitable choice of the frequency of oscillation of the point of suspension [1, 2] or by controlling its horizontal displacement [3]. The principle for ensuring the equilibrium of an inverted physical pendulum by controlling the angular acceleration of a flywheel fixed to the pendulum using a cylindrical hinge has also been investigated [4–7], where the required rotation of the flywheel is produced by means of an electric motor fixed to the pendulum.

Unlike the above-mentioned mechanical systems, the robot which is considered below does not have a directly hinged joint with the fixed, horizontal supporting plane but interacts with it through a horizontally-arranged, supporting circular cylinder. The robot is supported by the cylinder from below through a flat plate which ensures that there is no slip of the robot with respect to the cylinder and precludes the possibility of any tilting of the robot in the direction of the axis of the supporting cylinder. As a result, only plane-parallel motion of the robot in a vertical plane, perpendicular to the axis of the cylinder, is possible. In turn, there cannot be any slip of the cylinder on the supporting plane. The flywheel is a centrosymmetric body which is fixed at the centre of symmetry to the robot using a cylindrical hinge with an axis parallel to the axis of the supporting cylinder. The flywheel is driven by an electric motor fixed to the frame of the robot. Control is achieved by the electrical voltage on the windings of the motor. This system has three degrees of freedom, one of which (the angle of rotation of the flywheel) is directly controlled by the electric motor, while the other two (the angle of inclination of the robot to the horizontal plane and its rolling over the supporting cylinder) are unstable.

1. THE EQUATIONS OF MOTION

The phase DE is supported by uniform, horizontal right circular cylinder, the cross-section of which in the plane of the sketch is a circle with its centre at the point G (Fig. 1). The supporting cylinder G has a radius R, a mass m_0 and a central moment of inertia J_0. It lies on the horizontal plane and can roll along it without slip. The plane DE can roll along the supporting cylinder without slip. A rod is rigidly fixed perpendicular to the plate at the point O'. A flywheel of mass m, which can be rotated through an angle φ with respect to the rod, is fixed to the rod at the point B, which is a distance l from the plate. The centre of mass of the flywheel is located at the point B and its central moment of inertia is equal to J_m. The flywheel is set in motion by an electric drive. The reduction coefficient from the flywheel to the rotor of the electric motor is equal to γ and the moment of inertia of the rotor is equal to J_u. The overall centre of mass of the plate DE and the rod $O'B$, together with any components fixed to them, lies at the point C and the rod $O'B$. The distance $O'C$ is equal to a. The overall mass of the plate DE

†*Prikl. Mat. Mekh.* Vol. 67, No. 4, pp. 603–619, 2003.

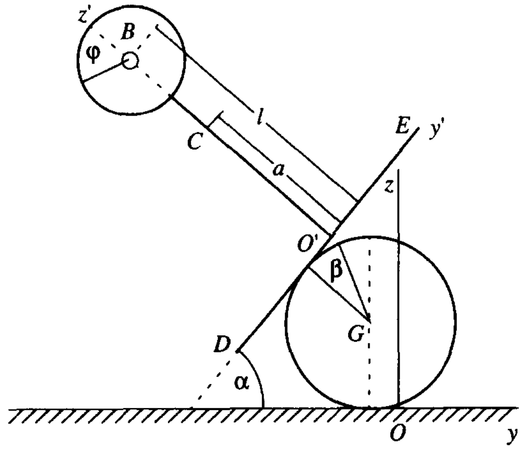

Fig. 1

and the rod $O'B$, together with the components fixed to them, is equal to M and their total central moment of inertia is equal to J

The angle between the plane of the plate DE and the supporting horizontal plane is denoted by α. We now consider the cross-section of the supporting cylinder in the plane of the sketch (Fig. 1). This cross-section is bounded by the circle $\mathcal{L}$ with its centre at the point G. Suppose β is the angle between the direction from G to a certain fixed point of the circumference of $\mathcal{L}$ and the direction from G to the point of contact of the segment DE with this circumference. We shall assume that, when $\alpha = \beta = 0$, the rod $O'B$, which is held in a vertical position, is projected onto the point of contact of the cylinder with the plane. The origin O of the fixed system of coordinates Oyz coincides with the point of support of the cylinder on the horizontal plane when $\alpha = \beta = 0$. The Oy axis is directed horizontally and the Oz axis vertically as shown in Fig. 1. In the system of coordinate $O'y'z'$, associated with rotor, the $O'y'$ axis is directed along the segment DE towards the point E and the $O'z'$ axis is directed along the segment $O'B$ towards the point B.

On taking account of the notation introduced, the absolute coordinates (y_0, z_0) of the point O', (y_b, z_b) of the point B and (y_c, z_c) of the point C can be expressed by the formulae

$$\begin{aligned} y_0 &= -R[\alpha + \sin\alpha - \beta(1+\cos\alpha)], \quad z_0 = R(1+\cos\alpha + \beta\sin\alpha) \\ y_b &= y_0 - l\sin\alpha, \quad z_b = z_0 + l\cos\alpha \\ y_c &= y_0 - a\sin\alpha, \quad z_c = z_0 + a\cos\alpha \end{aligned} \tag{1.1}$$

On taking into account that $\psi = \alpha - \beta$ is the angle of rotation of the supporting cylinder, the kinetic energy of the system can be represented as follows:

$$T = \frac{1}{2}(a'_{\alpha\alpha}\dot{\alpha}^2 + a'_{\beta\beta}\dot{\beta}^2 + a_{\varphi\varphi}\dot{\varphi}^2 - 2a'_{\alpha\beta}\dot{\alpha}\dot{\beta} + 2a_{\alpha\varphi}\dot{\alpha}\dot{\varphi}) \tag{1.2}$$

where

$$\begin{aligned} a'_{\alpha\alpha} &= 2b(1+\cos\alpha) + c(\beta^2 + 2\beta\sin\alpha) + d \\ a'_{\beta\beta} &= 2c(1+\cos\alpha) + e, \quad a_{\varphi\varphi} = J_m + \gamma^2 J_u \\ a'_{\alpha\beta} &= k(1+\cos\alpha) + c\beta\sin\alpha + e, \quad a_{\alpha\varphi} = J_m + \gamma J_u \\ b &= R(M+m)(R+z'_c), \quad c = (M+m)R^2 \\ d &= Ma^2 + ml^2 + J + J_m + J_u + e, \quad e = J_0 + m_0 R^2 \\ k &= R(M+m)(2R+z'_c), \quad z'_c = (aM+lm)/(M+m) \end{aligned} \tag{1.3}$$

The power function of the system has the form

$$\bar{U} = -\frac{g}{R}[b\cos\alpha + c(1+\beta\sin\alpha)] \tag{1.4}$$

where g is the acceleration due to gravity.

We now select the angles α, β and φ as Lagrangian coordinates and set up the system of Lagrange differential equations of the second kind

$$\begin{aligned} a'_{\alpha\alpha}\ddot{\alpha} - a'_{\alpha\beta}\ddot{\beta} &= -a_{\alpha\varphi}\ddot{\varphi} + I\left(\dot{\alpha}^2 + \frac{g}{R}\right) - 2c(\beta + \sin\alpha)\dot{\alpha}\dot{\beta} \\ -a'_{\alpha\beta}\ddot{\alpha} + a'_{\beta\beta}\ddot{\beta} &= (c\beta - I)\dot{\alpha}^2 + 2c\dot{\alpha}\dot{\beta}\sin\alpha - \frac{gc}{R}\sin\alpha \\ a_{\alpha\varphi}\ddot{\alpha} + a_{\varphi\varphi}\ddot{\varphi} &= Q, \quad I = b\sin\alpha - c\beta\cos\alpha \end{aligned} \tag{1.5}$$

Here Q is the generalized force which operates on changing the angle φ.

The function

$$f(\gamma) = \frac{a_{\alpha\varphi}}{a_{\varphi\varphi}} = \frac{j+\gamma}{j+\gamma^2}; \quad j = \frac{J_m}{J_u}$$

expresses the effect of the flywheel on the motion of the system. For $\gamma^* = -j$, we have $f(\gamma^*) = 0$. This is the unique root of the function $f(\gamma)$. When

$$\gamma = \gamma_1 = -j - \sqrt{j(1+j)}$$

the function $f(\gamma)$ reaches a minimum and, when

$$\gamma = \gamma_2 = -j + \sqrt{j(1+j)}$$

it reaches a maximum, whereupon

$$\gamma_1 < \gamma^* < 0 < \gamma_2, \quad \min f(\gamma) = f(\gamma_1) = \frac{1}{2\gamma_1}, \quad \max f(\gamma) = f(\gamma_2) = \frac{1}{2\gamma_2}$$

Moreover,

$$\lim_{\gamma \to +\infty} f(\gamma) = +0, \quad \lim_{\gamma \to -\infty} f(\gamma) = -0$$

The values of the maximum and minimum of $f(\gamma)$ can be controlled by means of a suitable choice of the moments of inertia of the flywheel and the rotor of the motor. For instance, if the ratio of the moment of inertia of the flywheel to the moment of inertia of the rotor of the motor is increased, then $f(\gamma) \to 1$ and $f(\gamma) \to -0$, and, theoretically $\min f(\gamma)$, can be made as small in absolute magnitude as desired.

When $\gamma^* < \gamma \leq \gamma_2$, the function $f(\gamma)$ increases monotonically from the value $f(\gamma^*) = 0$ up to the value $f(\gamma_2) > 1$. The extreme point γ^* of the indicated range of variation of γ has to be excluded since, when $\gamma = \gamma^*$, the effect of the control on the angles α and β of system (1.5) is lost.

By varying the magnitude of $\ddot{\varphi}$ in a suitable manner, it is possible to ensure different states of motion of the robot as a whole, We will now consider some special cases.

1. We require, for example, that the angle α should be constant during the whole time of motion ($\dot{\alpha} \equiv 0$). The first two equations of (1.5) take the form

$$-a'_{\alpha\beta}\ddot{\beta} = -a_{\alpha\varphi}\ddot{\varphi} + \frac{g}{R}I, \quad \ddot{\beta} = -\frac{gc\sin\alpha}{Ra'_{\beta\beta}}$$

The first of these equations serves as a matching condition for determining the function $\ddot{\varphi}$, and it follows from the second equation that, when $\alpha > 0$, the angle β will change in a uniformly retarded manner (the supporting cylinder in Fig. 1 moves with a constant positive angular acceleration, the plate DE has an acceleration which is directed downwards and the acceleration of the point of contact of the plate with the supporting cylinder, taken relative to the plate, is directed upwards). When $\alpha > 0$, the angle β will change in a uniformly accelerated manner (the supporting cylinder in Fig. 1 moves with a constant negative angular acceleration, the plate DE, as before, has an acceleration which is directed downwards, while the acceleration of the point of contact of the plate with the supporting cylinder, taken relative to the plate, is directed upwards).

Hence, by choosing suitable value of α, it is possible to ensure control of the relative position of the point of contact of the plate DE and the supporting cylinder. For example, if it is required to approach an indicated point of contact to the line BO' (Fig. 1) then, when $\beta > 0$ (the point of contact has a negative abscissa in $O'y'z'$ axes), it is necessary to choose $\alpha > 0$ but, in the case when $\beta < 0$ (the point of contact has a positive abscissa in the $O'y'z'$ axes), it is sufficient to take $\alpha < 0$.

2. If $\alpha = 0$, it is found that $\ddot{\beta} = 0$. This means that the plate DE, being horizontal, is uniformly displaced to the right or to the left with a constant initial velocity and the supporting cylinder executes a corresponding degenerate motion. In the case, the matching condition takes the form

$$\ddot{\varphi} = -\frac{gc\beta}{Ra_{\alpha\varphi}}$$

that is, the angular acceleration of the flywheel must be proportional to the magnitude of β and will be constant if β is constant. In particular, it will be equal to zero if $\beta = 0$.

3. It is clear from what has been said above that, when $Q = 0$, motion according to the law

$$\alpha \equiv 0, \quad \beta \equiv 0, \quad \dot{\varphi} \equiv \text{const}$$

will be steady motion of the system. An attempt to make this motion stable can be made by choosing the corresponding angular acceleration control $\ddot{\varphi}$.

4. The transition from one constant value of α to another constant value can be purposefully achieved by specifying the function $\ddot{\alpha}(t)$ in a special way. Such conditions in conjunction with the states corresponding to case 1 can turn out to be useful in ensuring stable motion of the system in the neighbourhood of a stationary point. If $\ddot{\alpha}(t)$ is a specified function of time, then, as previously, the first equation of (1.5) gives the matching condition and the second equation completely defines the variation of the angle β. Moreover, this equation can be rewritten in the form

$$\begin{aligned} a'_{\beta\beta}\ddot{\beta} &= c\beta[(1+\cos\alpha)\dot{\alpha}^2 + \ddot{\alpha}\sin\alpha] + \\ &+ \left(2c\dot{\alpha}\dot{\beta} - b\dot{\alpha}^2 - \frac{gc}{R}\right)\sin\alpha + [k(1+\cos\alpha) + e]\ddot{\alpha} \end{aligned} \tag{1.6}$$

It is found to be linear with respect to β with coefficients which depend on time in a known manner.

We will specify the variation of the angle α in a small neighbourhood of the values $\alpha = \dot{\alpha} = 0$ using the equation

$$\ddot{\alpha} + \omega^2\alpha = 0 \tag{1.7}$$

which denotes harmonic swinging of the plate DE about the horizontal position. This relation can be used for the transfer of the robot from one inclined position to another.

In Eq. (1.6), we take account of equality (1.7) and neglect terms greater than the second order infinitesimals in α, $\dot{\alpha}$, $\ddot{\alpha}$. We then obtain

$$(4c+e)\ddot{\beta} = -\frac{gc}{R}\alpha + (2k+e)\ddot{\alpha} \tag{1.8}$$

Suppose the values $\beta(t_0) = \beta_0$, $\dot{\beta}(t_0) = \dot{\beta}_0$, are realized at the instant of time t_0 of the start of the transfer of the plate from a position with an angle of inclination $\alpha(t_0) = \alpha_0$ and an angular velocity $\dot{\alpha}(t_0) = 0$. The corresponding equation of motion (1.8) takes the form

$$\beta - \beta_0 = \frac{1}{4c+e}\left(\frac{gc}{R\omega^2} + 2k + e\right)(\alpha - \alpha_0) + (t - t_0)\dot{\beta}_0 \tag{1.9}$$

We shall assume that the equalities $\alpha(t_1) = -\alpha_0$ and $\dot{\alpha}(t_1) = 0$ have to be ensured at the final instant of time t_1 of the transfer of the plate into the second inclined position. When account is taken of the fact that $t_1 - t_0 = \pi/\omega$, from relation (1.9) we find an approximate formula for the increment in the angle β during the time of the manoeuvre

$$\beta(t_1) - \beta_0 = -2\alpha_0\frac{1}{4c+e}\left(\frac{gc}{R\omega^2} + 2k + e\right) + \frac{\pi\dot{\beta}_0}{\omega} \tag{1.10}$$

It is clear that the effect of the term associated with the angular velocity $\dot{\beta}_0$ can be made as small as desired by increasing ω (reducing the transition time) while it is impossible to make the term containing α_0 as small as desired in this way. The amplitude of the manoeuvre with respect to α affects it, increasing the absolute magnitude of the inclination β. Revolving manoeuvres with too large an amplitude of the angle α may not always turn out to be acceptable on account of the brushing the plate against the floor.

The necessity for the manoeuvre indicated arises, for example, if, at a certain instant of time t_0, it turns out that

$$\alpha_0 > 0, \quad \beta_0 < 0, \quad \dot{\beta}_0 < 0$$

This means that the plate DE has descended too low and it can only get back to the equilibrium position by turning it over such that the values of α become negative. The case when

$$\alpha_0 < 0, \quad \beta_0 > 0, \quad \dot{\beta}_0 > 0$$

is analogous.

2. A LINEAR SYSTEM

We will now consider the equations of the linear approximation in the neighbourhood of the solution $\alpha = 0, \beta = 0$

$$\begin{aligned} &a_{\alpha\alpha}\ddot{\alpha} - a_{\alpha\beta}\ddot{\beta} + a_{\alpha\varphi}\ddot{\varphi} = \frac{g}{R}(b\alpha - c\beta), \quad -a_{\alpha\beta}\ddot{\alpha} + a_{\beta\beta}\ddot{\beta} = -\frac{gc}{R}\alpha \\ &a_{\varphi\varphi}\ddot{\varphi} + a_{\alpha\varphi}\ddot{\alpha} = Q \end{aligned} \tag{2.1}$$

where

$$a_{\alpha\alpha} = 4b + d, \quad a_{\alpha\beta} = 2(b+c) + e, \quad a_{\beta\beta} = 4c + e \tag{2.2}$$

Suppose the system is not controlled: $Q = 0$. One of the roots of the characteristic equations of the control system $\lambda_1 = 0$. The other two roots are found from the quadratic equation

$$\Delta_0\lambda^2 + \frac{ga_{\varphi\varphi}}{R}(2ca_{\alpha\beta} - ba_{\beta\beta})\lambda - \frac{g^2c^2a_{\varphi\varphi}}{R^2} = 0 \tag{2.3}$$

where

$$\begin{aligned} \Delta &= a_{\alpha\alpha}a_{\beta\beta} - a_{\alpha\beta}^2 = 4R^2[Mm(l-a)^2 + (M+m)(J + J_m + J_u)] + \\ &+ (Ma^2 + ml^2 + J + J_m + J_u)(J_0 + m_0R^2) \end{aligned}$$

$$\Delta_0 = a_{\varphi\varphi}\Delta - a_{\alpha\varphi}^2a_{\beta\beta}$$

Δ_0 is the determinant of the kinetic energy matrix of the system and Δ is the determinant of the first two equations of system (2.1), which is non-zero for any non-zero robot parameters.

We see that Δ_0 depends quadratically on the reduction coefficient γ:

$$\Delta_0 = J_u(\Delta - J_ua_{\beta\beta})\gamma^2 - 2J_mJ_ua_{\beta\beta}\gamma + J_m(\Delta - J_ma_{\beta\beta})$$

The discriminant $\mathcal{D}$ of this quadratic trinomial has the form

$$\mathcal{D} = -J_m J_u \Delta\{4R^2[Mm(l-a)^2 + (M+m)J] + (Ma^2 + ml^2 + J)e\} < 0$$

Hence, $\Delta_0 > 0$ for any values of the robot parameters. Consequently, Eq. (2.3) has a single negative root: $\lambda_2 < 0$ and a single positive root: $\lambda_3 > 0$. On the whole, when there is not control, system (2.1) has a single neutral principal coordinate which is obviously associated with the existence of the flywheel, a single stable principal coordinate and a single unstable coordinate. One of the characteristic exponents is positive, three of them have a real part equal to zero and one is negative. This distinguishes the system under consideration from those studied earlier [7].

System (2.1) can be solved for $\ddot{\alpha}$ and $\ddot{\beta}$ and reduced to the form

$$\ddot{\alpha} = -\frac{gca_{\alpha\beta}}{R\Delta}\alpha - \frac{a_{\beta\beta}}{\Delta}\bar{u}, \quad \ddot{\beta} = -\frac{gca_{\alpha\alpha}}{R\Delta}\alpha - \frac{a_{\alpha\beta}}{\Delta}\bar{u} \tag{2.4}$$

where $\bar{u}$ is expressed by the formula

$$\bar{u} = a_{\alpha\varphi}\ddot{\varphi} - \frac{gb}{R}\alpha + \frac{gc}{R}\beta \tag{2.5}$$

and can, at this stage of the investigation, be considered as the control of the system.

3. SYNTHESIS OF THE CONTROL LAW

Taking $\bar{u}$ as the control, we will investigate the properties of the controllability of system (2.4), (2.5) with respect to the variables α, β and $\dot{\varphi}$. For this purpose, we will represent it in the standard form

$$\dot{\mathbf{x}} = \mathcal{A}\mathbf{x} + \mathcal{B}\bar{u} \tag{3.1}$$

where

$$\mathbf{x} = \begin{Vmatrix} \alpha \\ \beta \\ \dot{\alpha} \\ \dot{\beta} \\ \dot{\varphi} \end{Vmatrix}, \quad \mathcal{A} = \begin{Vmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ a_{31} & 0 & 0 & 0 & 0 \\ a_{41} & 0 & 0 & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 & 0 \end{Vmatrix}, \quad \mathcal{B} = \begin{Vmatrix} 0 \\ 0 \\ b_3 \\ b_4 \\ b_5 \end{Vmatrix}$$

$$a_{31} = -\frac{gca_{\alpha\beta}}{R\Delta}, \quad a_{41} = -\frac{gca_{\alpha\alpha}}{R\Delta}, \quad a_{51} = \frac{gb}{Ra_{\alpha\varphi}}, \quad a_{52} = -\frac{gc}{Ra_{\alpha\varphi}}$$

$$b_3 = -\frac{a_{\beta\beta}}{\Delta}, \quad b_4 = -\frac{a_{\alpha\beta}}{\Delta}, \quad b_5 = \frac{1}{a_{\alpha\varphi}}$$

For the controllability matrix $\mathcal{U} = (\mathcal{B}, \mathcal{A}\mathcal{B}, \mathcal{A}^2\mathcal{B}, \mathcal{A}^3\mathcal{B}, \mathcal{A}^4\mathcal{B})$ [8], we find

$$\det\mathcal{U} = -b_3^2 a_{52}(b_3 a_{41} - b_4 a_{31})^3$$

where

$$b_3 a_{41} - b_4 a_{31} = \frac{gc}{R\Delta^2}(a_{\beta\beta}a_{\alpha\alpha} - a_{\alpha\beta}^2) = \frac{gc}{R\Delta} > 0$$

Consequently, in the case of an unbounded $\bar{u}$, system (3.1) possesses the property of total controllability.

We shall take $\bar{u}$ in the form [9]

$$\bar{u} = \kappa_1\alpha + \kappa_2\beta + \kappa_3\dot{\alpha} + \kappa_4\dot{\beta} + \kappa_5\dot{\varphi} \tag{3.2}$$

such that the equilibrium position becomes asymptotically stable with respect to the coordinates α, β and φ. The linear system (3.1) is then written as follows:

$$\begin{aligned}\ddot{\alpha} &= (a_{31}+\kappa_1 b_3)\alpha+\kappa_2 b_3\beta+\kappa_3 b_3\dot{\alpha}+\kappa_4 b_3\dot{\beta}+\kappa_5 b_3\dot{\varphi}\\ \ddot{\beta} &= (a_{41}+\kappa_1 b_4)\alpha+\kappa_2 b_4\beta+\kappa_3 b_4\dot{\alpha}+\kappa_4 b_4\dot{\beta}+\kappa_5 b_4\dot{\varphi}\\ \ddot{\varphi} &= (a_{51}+\kappa_1 b_5)\alpha+(a_{52}+\kappa_2 b_5)\beta+\kappa_3 \beta_5\dot{\alpha}+\kappa_4 b_5\dot{\beta}+\kappa_5 b_5\dot{\varphi}\end{aligned} \tag{3.3}$$

and the corresponding characteristic equation takes the form

$$\lambda^5+a_1\lambda^4+a_2\lambda^3+a_3\lambda^2+a_4\lambda+a_5 = 0 \tag{3.4}$$

where

$$\begin{aligned}a_1 &= c_{13}\kappa_3+c_{14}\kappa_4+c_{15}\kappa_5, \quad a_2 = c_{21}\kappa_1+c_{22}\kappa_2+c_{20}\\ a_3 &= c_{34}\kappa_4+c_{35}\kappa_5, \quad a_4 = c_{42}\kappa_2, \quad a_5 = c_{55}\kappa_5\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}c_{13} &= c_{21} = -b_3 = \frac{a_{\beta\beta}}{\Delta}, \quad c_{14} = c_{22} = -b_4 = \frac{a_{\alpha\beta}}{\Delta}\\ c_{20} &= -a_{31} = \frac{gca_{\alpha\beta}}{R\Delta}, \quad c_{15} = -b_5 = -\frac{1}{a_{\alpha\varphi}}\\ c_{34} &= c_{42} = a_{31}b_4-a_{41}b_3 = -\frac{gc}{R\Delta}, \quad c_{55} = a_{52}c_{34} = \frac{g^2c^2}{R^2\Delta a_{\alpha\varphi}}\\ c_{35} &= a_{31}b_5-a_{52}b_4-a_{51}b_3 = \frac{g(be-2ce-4c^2)}{R\Delta a_{\alpha\varphi}}\end{aligned} \tag{3.6}$$

On solving equalities (3.5) for the coefficients κ_i, we find

$$\begin{aligned}\kappa_1 &= d_{12}a_2+d_{14}a_4-d_{10}, \quad \kappa_2 = -d_{24}a_4, \quad \kappa_3 = d_{31}a_1+d_{33}a_3+d_{35}a_5\\ \kappa_4 &= -d_{43}a_3+d_{45}a_5, \quad \kappa_5 = d_{55}a_5\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}d_{12} &= d_{31} = \frac{\Delta}{a_{\beta\beta}}, \quad d_{14} = d_{33} = \frac{Ra_{\alpha\beta}\Delta}{gca_{\beta\beta}}, \quad d_{10} = \frac{gca_{\alpha\beta}}{Ra_{\beta\beta}}\\ d_{24} &= d_{43} = \frac{R\Delta}{gc}, \quad d_{35} = \frac{R^2\Delta}{g^2c^2a_{\beta\beta}}\left[\Delta-\frac{(be-2ce-4c^2)a_{\alpha\beta}}{c}\right]\\ d_{45} &= \frac{(be-2ce-4c^2)R^2\Delta}{g^2c^3}, \quad d_{55} = \frac{R^2a_{\alpha\varphi}\Delta}{g^2c^2}\end{aligned} \tag{3.8}$$

In order to guarantee the asymptotic stability of the transient, it is sufficient to choose the roots λ_i of the characteristic equation (3.4) as follows:

$$\lambda_1<\lambda_2<\lambda_3<\lambda_4<\lambda_5<0 \tag{3.9}$$

The coefficients of the characteristic equation with these roots are found using Newton's binomial rule

$$\begin{aligned}a_1 &= -\sum_{j=1}^{5}\lambda_j, \quad a_2 = \sum_{i=1}^{4}\lambda_i\sum_{j=i+1}^{5}\lambda_j, \quad a_3 = -\sum_{i=1}^{3}\lambda_i\sum_{j=i+1}^{4}\lambda_j\sum_{k=j+1}^{5}\lambda_k\\ a_4 &= \sum_{i=1}^{2}\lambda_i\sum_{j=i+1}^{3}\lambda_j\sum_{k=j+1}^{4}\lambda_k\sum_{n=k+1}^{5}\lambda_n, \quad a_5 = -\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\end{aligned} \tag{3.10}$$

and the coefficients of the control law (3.2) can be calculated using formulae (3.7).

4. ELECTRIC DRIVE CONTROL

We will now consider the third equation of system (2.1). In this equation, the right-hand side is the moment of the electromagnetic forces applied to the rotor of the electric motor. We take [10] the approximate value of the quantity Q

$$Q = c_1 u - c_2 \dot{\varphi} \tag{4.1}$$

The constants $c_1 > 0$ and $c_2 > 0$ are found using the technical and operational data for the drive, allowing for the reduction coefficient. The electrical voltage is bounded in magnitude: $u_0 \leq u \leq u_0$.

On substituting the quantity $\ddot{\alpha}$ taken from the first equation of (2.4) and the quantity $\ddot{\varphi}$ found from Eq. (2.5) into the third equation of system (2.1), we establish the link between u and $\bar{u}$

$$A\bar{u} + B\alpha - C\beta = c_1 u - c_2 \dot{\varphi} \tag{4.2}$$

where

$$A = \frac{\Delta_0}{a_{\alpha\varphi}\Delta}, \quad B = \frac{g}{R}\left(\frac{ba_{\varphi\varphi}}{a_{\alpha\varphi}} - \frac{ca_{\alpha\beta}a_{\alpha\varphi}}{\Delta}\right), \quad C = \frac{gca_{\varphi\varphi}}{Ra_{\alpha\varphi}}$$

The determinant $\Delta_0 > 0$ occurs in the numerator of the expression of the coefficient A, and therefore $A > 0$ for any values of the system parameters.

We substitute expression (3.2) for $\bar{u}$ into Eq. (4.2) and use the notation

$$V = k_1\alpha + k_2\beta + k_3\dot{\alpha} + k_4\dot{\beta} + k_5\dot{\varphi} \tag{4.3}$$

where

$$k_1 = \frac{A\kappa_1 + B}{c_1}, \quad k_2 = \frac{A\kappa_2 - C}{c_1}, \quad k_3 = \frac{A\kappa_3}{c_1}$$
$$k_4 = \frac{A\kappa_4}{c_1}, \quad k_5 = \frac{A\kappa_5 + c_2}{c_1} \tag{4.4}$$

We now specify the law for the voltage fed to the motor

$$u = \begin{cases} -u_0, & V < -u_0 \\ u = V, & |V| \leq u_0 \\ u_0, & V > u_0 \end{cases} \tag{4.5}$$

The occurrence of some kind of non-zero amplification factor k_i in expression (4.3) indicates the need to measure the corresponding phase coordinate. We will now analyse the possibility that each of the coefficients k_i vanishes. We start from the coefficient k_1. When account is taken of the positiveness of a_2 and a_4, it follows from the first formula of (4.4) and the expressions for κ_1 in formulae (3.7) and for the coefficients d_{12} and d_{14} in (3.8) that the possibility of the coefficient k_1 vanishing depends on the sign of the expression $Ad_{10} + B$. On carrying out a transformation taking account of relations (2.2), we find

$$-Ad_{10} + B = \frac{ga_{\varphi\varphi}(ba_{\beta\beta} - ca_{\alpha\beta})}{Ra_{\alpha\varphi}a_{\beta\beta}} = \frac{ga_{\varphi\varphi}(b-c)(2c+e)}{Ra_{\alpha\varphi}a_{\beta\beta}} > 0$$

It is clear from this that, in the case of real roots λ_i which satisfy (3.9), the coefficient k_1 cannot vanish for any values of the system parameters. Similarly, since $a_4 > 0$, k_2 also cannot vanish.

As far as the coefficients k_3 and k_4 are concerned, they cannot simultaneously be made equal to zero. We will now show that this is so. We will assume that the condition $k_3 = k_4 = 0$ is satisfied. This is equivalent to the system of equations

$$d_{31}a_1 + d_{33}a_3 + d_{35}a_5 = 0, \quad -d_{43}a_3 + d_{45}a_5 = 0$$

Expressing a_3 from the second equation of this system and substituting it into the first equation, we obtain

$$d_{31}a_1 + \left(\frac{d_{45}d_{33}}{d_{43}} + d_{35}\right)a_5 = d_{31}a_1 + \frac{R^2\Delta^2}{g^2c^2a_{\beta\beta}}a_5$$

All the coefficients in this last expression are found to be positive and it cannot vanish. It is therefore necessary to consider the cases when either $k_3 = 0$ or $k_4 = 0$.

The coefficient b in the expression for d_{45} in formulae (3.8) depends on the coordinate z_c' of the centre of mass of the robot (see (1.3)). If the value of z_c' is small (a short robot), then $d_{45} < 0$. But, then, $d_{35} > 0$ and neither of the coefficients k_3, k_4 can vanish. We now use the notation

$$\bar{z}_c' = R\left[1 + \frac{4R^2(M+m)}{J_0 + m_0R^2}\right]$$

If $z_c' = \bar{z}_c'$, then $d_{45} = 0$ and at the same time it is found that $d_{35} > 0$. Consequently, when $z_c' = \bar{z}_c'$, it is impossible for the coefficients k_3 and k_4 to be equal to zero as previously. When there is an increase in $z_c' > \bar{z}_c'$, the coefficient d_{45} becomes positive and it is now possible to make the coefficient k_4 equal to zero. Hence, the condition

$$z_c' > \bar{z}_c' \tag{4.6}$$

constrains the height of the robot from below and enables one to avoid having to measure the parameter $\dot{\beta}$.

Finally, the coefficient k_5 cannot vanish on account of the fact that $a_5 > 0$ and $d_{55} > 0$.

Suppose condition (4.6) is satisfied. Then, the roots of the characteristic equation (3.4) can be selected in the following way. We arbitrarily take five real numbers $\bar{\lambda}_i$ which satisfy the condition

$$\bar{\lambda}_1 < \bar{\lambda}_2 < \bar{\lambda}_3 < \bar{\lambda}_4 < \bar{\lambda}_5 < 0 \tag{4.7}$$

and, using formulae (3.10), we find the values of $\bar{a}_3$ and $\bar{a}_5$ corresponding to them. We choose the roots of the characteristic equation (3.4) in the form

$$\lambda_i = \chi\bar{\lambda}_i, \quad i = 1, \dots, 5$$

The requirement that $k_4 = 0$ and $\chi \neq 0$ leads to the equation

$$\chi^2 d_{45}\bar{a}_5 - d_{43}\bar{a}_3 = 0$$

Whereupon we find that

$$\chi = \sqrt{\frac{d_{43}\bar{a}_3}{d_{45}\bar{a}_5}} \tag{4.8}$$

After carrying out the procedure indicated above we have

$$k_1c_1 > 0, \quad k_2c_1 < 0, \quad k_3c_1 > 0, \quad k_4 = 0, \quad k_5c_1 > 0$$

5. THE DOMAIN OF STABILIZABILITY

The control with saturation (4.5) implies a contraction of the domain of controllability. We will now investigate the possibility of control with respect to the coordinates α and β. In order to determine the corresponding domain of controllability, we take the system of equations (2.1) and substitute the expression for the generalized force (4.1) into it. We then eliminate $\ddot{\varphi}$ from the first equation and consider the first two equations of the system

$$\left(a_{\alpha\alpha} - \frac{a_{\alpha\varphi}^2}{a_{\varphi\varphi}}\right)\ddot{\alpha} - a_{\alpha\beta}\ddot{\beta} = \frac{g}{R}(b\alpha - c\beta) + U, \quad -a_{\alpha\beta}\ddot{\alpha} + a_{\beta\beta}\ddot{\beta} = -\frac{gc}{R}\alpha \tag{5.1}$$

where

$$U = a_{\alpha\varphi}(c_2\dot{\varphi} - c_1u)/a_{\varphi\varphi} \tag{5.2}$$

At each fixed instant of time the control U can take values from the range

$$U_- \leq U \leq U_+, \quad U_\pm = a_{\alpha\varphi}(c_2\dot{\varphi} \pm |c_1|u_0)/a_{\varphi\varphi} \tag{5.3}$$

We shall understand the controllability of system (5.1) to be the possibility of bringing its phase point to the origin of the coordinates of phase space, that is, to the point with coordinates $\alpha = \dot{\alpha} = \beta = \dot{\beta} = 0$. In order to investigate the stabilizability, we will consider the properties of a homogeneous system which corresponds to system (5.1). Its characteristic equation is identical to (2.3) and has one positive and one negative root. We note that, if the coefficient d_{45} in formulae (3.8) is positive, then the positive root will be the greater in absolute magnitude but, if d_{45} turns out to be negative, then the negative root will be the greater in absolute magnitude. However, there are no system parameters for which any of the roots is equal to zero. We denote the roots of Eq. (2.3) by $\mu_1 < 0$, $\mu_2 > 0$.

The eigenvectors of the homogeneous system being considered have the coordinates

$$\alpha_1 = \mu_1 a_{\beta\beta}, \quad \beta_1 = \mu_1 a_{\alpha\beta} - gc/R$$
$$\alpha_2 = \mu_2 a_{\beta\beta}, \quad \beta_2 = \mu_2 a_{\alpha\beta} - gc/R$$

After transforming the coordinates to eigenvectors, system (5.1) changes into the following system

$$\ddot{\xi}_1 = \mu_1\xi_1 + u_1, \quad \ddot{\xi}_2 = \mu_2\xi_2 + u_2 \tag{5.4}$$

where the controls

$$u_1 = U\alpha_1 = U\mu_1 a_{\beta\beta}, \quad u_2 = U\alpha_2 = U\mu_2 a_{\beta\beta}$$

are bounded in magnitude:

$$u_{1-} \le u_1 \le u_{1+}, \quad u_{2-} \le u_2 \le u_{2+}$$

where

$$u_{1\pm} = U_{\mp}\mu_1 a_{\beta\beta}, \quad u_{2\pm} = U_{\pm}\mu_2 a_{\beta\beta} \tag{5.5}$$

The first equation of system (5.4) corresponds to a negative eigenvalue μ_1. In the case of constant control, these phase trajectories are ellipses. From any point in phase space, it is possible to transfer the phase point of the first equation to the origin of its phase plane by selecting a control in the range $u_{1-} \leqslant u_1 \leqslant u_{1+}$ in a suitable manner.

The second equation of (5.4) corresponds to the positive eigenvalue μ_2. For the constant control u_2, its phase trajectories are hyperbolae with asymptotes

$$\dot{\xi}_2 = \pm\sqrt{\mu_2}(\xi_2 - \bar{u}_2); \quad \bar{u}_2 = -u_2/\mu_2$$

We take any numbers u_- and u_+ such that

$$u_- < 0 < u_+$$

and investigate the possibility of transferring the system described by the equation

$$\ddot{\xi} - \mu_2\xi = u, \quad u_- \le u \le u_+ \tag{5.6}$$

from different initial phase points to the origin of the phase plane using a bang-bang control of the form

$$u = \frac{u_- + u_+}{2} \pm \frac{u_- - u_+}{2} \tag{5.7}$$

If we put $u = u_-$, the phase hyperbolae for (5.6) are given by the equation

$$\mu_2(\xi - \bar{u}_-)^2 - \dot{\xi}^2 = p_-, \quad \bar{u}_- = -u_-/\mu_2 \tag{5.8}$$

but, if we put $u = u_+$, the analogous equation takes the form

$$\mu_2(\xi - \bar{u}_+)^2 - \dot{\xi}^2 = p_+, \quad \bar{u}_+ = -u_+/\mu_2 \tag{5.9}$$

where p_- and p_+ are integration constants and $\bar{u}_+ < 0 < \bar{u}_-$.

Initially, we separate out the domain of initial conditions which satisfy the inequalities

$$-\sqrt{\mu_2}(\xi_0 - \bar{u}_-) \le \dot{\xi}_0 < 0 \tag{5.10}$$

If the initial point lies in this domain and the control $u = u_-$ is maintained, then motion of the phase point occurs along the corresponding hyperbola (5.8) towards the abscissa, and the phase trajectory intersects the abscissa at a point with the coordinate

$$\xi_- = [(\xi_0 - \bar{u}_-)^2 - \dot{\xi}_0^2/\mu_2]^{1/2} + \bar{u}_- > \bar{u}_- > 0$$

and then departs to infinity. If the control $u = u_+$ is taken with the same initial conditions, motion will also take place towards the abscissa, but now along a hyperbola of the form of (5.9), and the coordinate ξ_+ of the point of intersection of this hyperbola with the abscissa is expressed as follows

$$\xi_+ = [(\xi_0 - \bar{u}_+)^2 - \dot{\xi}_0^2/\mu_2]^{1/2} + \bar{u}_+$$

It can be shown† that

$$\xi_+ \geq \xi_- \tag{5.11}$$

Inequality (5.11) confirms the fact that it is impossible to transfer a phase trajectory which starts out in the domain (5.10) to the origin of the coordinates for any control $u \in [u_-, u_+]$.

Trajectories which begin in the domain described by the inequalities

$$\dot{\xi}_0 \geq -\sqrt{\mu_2}(\xi_0 - \bar{u}_-), \quad \xi_0 \geq 0$$

also cannot be transferred to the origin of coordinates. For any control $u \in [u_-, u_+]$, they depart to infinity.

Similarly, the domain of initial conditions

$$-\sqrt{\mu_2}(\xi_0 - \bar{u}_+) > \dot{\xi}_0$$

does not satisfy the requirement of stabilizability.

We will now show that it is possible to construct a bang-bang control (5.7) which transfers the phase trajectory to the origin of coordinates from any initial point in the domain

$$-\sqrt{\mu_2}(\xi_0 - \bar{u}_+) < \dot{\xi}_0 < -\sqrt{\mu_2}(\xi_0 - \bar{u}_-) \tag{5.12}$$

For this purpose we formulate the switching line

$$\dot{\xi}_s(\xi) = \begin{cases} \sigma_-, & \xi < 0 \\ -\sigma_+, & \xi \geq 0 \end{cases}, \quad \sigma_\pm = \{\mu_2[(\xi - \bar{u}_\pm)^2 - \bar{u}_\pm^2]\}^{1/2} \tag{5.13}$$

and synthesize the equation which transfers any phase point from the domain (5.12) to the origin of coordinates

$$u = \begin{cases} u_+, & -\sqrt{\mu_2}(\xi_0 - \bar{u}_+) < \dot{\xi} < \dot{\xi}_s \\ u_+, & \dot{\xi} = \dot{\xi}_s \text{ and } \dot{\xi}_s < 0 \\ u_-, & -\sqrt{\mu_2}(\xi_0 - \bar{u}_-) > \dot{\xi} > \dot{\xi}_s \\ u_-, & \dot{\xi} = \dot{\xi}_s \text{ and } \dot{\xi}_s \geq 0 \end{cases}$$

Hence, the system described by Eq. (5.6) with the constant bounds u_- and u_+, can be stabilized if and only if its phase point belongs to the domain (5.12).

The bang-bang control which has been presented only proves that it is possible to stabilize the system. In the case of an actual control, it is advisable to use functions $u(t)$ which do not reach the boundaries of the domain of permissible values.

The second equation of (5.4) differs from Eq. (5.6) which has been considered in that the bounds u_{2-} and u_{2+} are not constant but depend on the angular velocity $\dot{\varphi}$. When $\dot{\varphi}$ changes, the range of permissible values of the control function is shifted as a whole in one direction or the other depending on the sign of $\dot{\varphi}$.

†GOLUBEV, Yu. F., A Robot – Balancer. Preprint No. 50. Inst. Prikl. Mat., Moscow, 2002.

We now indicate the boundaries of the domain of stabilizability for the case when the flywheel is not rotating. They are of interest in relation to the fact that bringing the system into an equilibrium position, subject to the condition that the angular velocity of the flywheel tends to zero, serves as the aim of the control. In formulae (5.5), we put $\dot{\varphi} = 0$. Then

$$U_- = -U_+ = -\frac{a_{\alpha\varphi}|c_1|u_0}{a_{\varphi\varphi}}, \quad u_{2-} = -u_{2+} = -\frac{a_{\alpha\varphi}|c_1|u_0}{a_{\varphi\varphi}}\mu_2 a_{\beta\beta}$$

Now, on making use of inequality (5.12) and changing from the principal to the initial coordinates, we find the domain of stabilizability of system (5.1)

$$-u_0 < a_\beta\left(\beta + \frac{\dot{\beta}}{\sqrt{\mu_2}}\right) - a_\alpha\left(\alpha + \frac{\dot{\alpha}}{\sqrt{\mu_2}}\right) < u_0 \tag{5.14}$$

where

$$a_\beta = \frac{\mu_1 R a_{\varphi\varphi}}{(\mu_2 - \mu_1) g c |c_1| a_{\alpha\varphi} a_{\beta\beta}}, \quad a_\alpha = \frac{a_\beta}{a_{\beta\beta}}\left(a_{\alpha\beta} - \frac{gc}{\mu_1 R}\right)$$

6. THE EFFECT OF ERRORS

The measurement errors depend on the sensors used in the system. We shall assume that the measurements are made using a fixed telecamera located in the frontal plane outside the robot which regards the position of some section rigidly associated with the robot. Let this section be a part of the straight line $O'B$ shown in Fig. 1. Then, in the axes associated with the robot, the terminal points of this section N_1 and N_2, should have coordinates $(0, z_1')$ and $(0, z_2')$ $(z_2' > z_1' > 0)$, respectively. We choose the Ox axis to complete the system of coordinates Oyz to a right-handed trihedral. Assuming that the coordinates of the points N_1 and N_2 are measured without errors and shifting the origin of the coordinates into the picture plane of the telecamera, we find, in accordance with formulae (1.1), the absolute coordinates of the ends of the section being measured

$$y_n = y_0 - z_n'\sin\alpha, \quad z_n = z_0 + z_n'\cos\alpha, \quad x_n = -h, \quad n = 1, 2 \tag{6.1}$$

where h is the distance from the picture plane of the telecamera to the plane of the motion. The picture plane must be placed strictly parallel to the plane of motion of the robot. In reality, it will be set up with errors, the action of which is expressed by the matrix of small rotations

$$\Omega = \begin{Vmatrix} 1 & -\delta_3 & \delta_2 \\ \delta_3 & 1 & -\delta_1 \\ -\delta_2 & \delta_1 & 1 \end{Vmatrix}$$

As a result, the images of the points N_n $(n = 1, 2)$ in the matrix of the telecamera will have the following points respectively

$$\begin{aligned} y_n'' &= k[y_0 - z_n'\sin\alpha + \delta_3 h + \delta_2(z_0 + z_n'\cos\alpha)] \\ z_n'' &= k[-\delta_2(y_0 - z_n'\sin\alpha) - \delta_1 h + z_0 + z_n'\cos\alpha], \quad n = 1, 2 \end{aligned} \tag{6.2}$$

where k is the contraction coefficient of the image in the telecamera matrix.

The subsequent analysis of the structure of the errors must rely on the actual processing algorithm taking account of the discretization of the image. Here, we shall assume that the images of the points N_n $(n = 1, 2)$ in the telecamera matrix are correctly identified and we take the simplest algorithm for determining the angles α and β. From relations (6.2), we find

$$\frac{y_2'' - y_1''}{z_2'' - z_1''} = \frac{-\sin\alpha + \delta_2\cos\alpha}{\cos\alpha + \delta_2\sin\alpha} \approx -\operatorname{tg}\alpha + \delta_2(1 - \operatorname{tg}\alpha) \tag{6.3}$$

Hence, tgα will be calculated with constant and proportional errors and, moreover, as would be expected, the constant error and the coefficient of proportionality were found to be equal to the angle of rotation of the telecamera about the perpendicular to the plane of motion.

It is clear from formula (1.1) that the angle β can be reliably found even for small angles α by calculating the quantity y_0. We shall use formulae (6.2) when $n = 2$

$$y_0 + \delta_3 h + \delta_2 z_0 = \frac{y_1'' z_2' - y_2'' z_1'}{k(z_2' - z_1')} \tag{6.4}$$

If the resulting expression is compared with the first formula of (1.1), it can be seen that, when determining the angle β, additional constant and proportional errors also arise apart from the errors associated with the determination of the angle α. One of these arises due to the rotation of the telecamera about the vertical axis and is proportional to the ratio of the distance from the robot to the telecamera to the radius of the supporting cylinder. A second constant error is associated with the rotation of the telecamera about an axis perpendicular to the plane of motion.

Formulae (6.3) and (6.4) can be proposed as a basis for aligning the position of the telecamera. However, even after this, certain random and systematic constant and proportional errors remain when determining the angles α and β. In order to elucidate the effect of systematic errors on the control process, we assume that the structure of the errors is linear with constant coefficients

$$\delta\alpha = \delta_\alpha + \delta_\alpha'\alpha, \quad \delta\dot{\alpha} = \bar{\delta}_\alpha + \bar{\delta}_\alpha'\dot{\alpha}$$

$$\delta\beta = \delta_\beta + \delta_\beta'\beta, \quad \delta\dot{\beta} = \bar{\delta}_\beta + \bar{\delta}_\beta'\dot{\beta}$$

$$\delta\dot{\varphi} = \bar{\delta}_\varphi + \bar{\delta}_\varphi'\dot{\varphi}$$

The chosen control law is such that proportional errors lead to a certain change in the amplification factors and, if the amplification factors are taken with sufficient margin, the effect of the proportional errors will be insignificant.

The existence of constant errors leads to a state of affairs when an additional constant term arises in the last equation of system (2.1) and this equation takes the form

$$a_{\varphi\varphi}\ddot{\varphi} + a_{\alpha\varphi}\ddot{\alpha} = c_1 u + c_1\delta u - c_2\dot{\varphi}$$

$$\delta u = k_1\delta_\alpha + k_2\bar{\delta}_\alpha + k_3\delta_\beta + k_4\bar{\delta}_\beta + k_5\bar{\delta}_\varphi$$

Then, system (2.1) admits of the particular solution

$$\dot{\varphi} = c_1\delta u/c_2, \quad \alpha \equiv 0, \quad \beta \equiv 0 \tag{6.5}$$

as a consequence of which the control ensures the asymptotic stability of the equilibrium with respect to the angles α and β, and the term (6.5), which compensates for the action of the constant components of the errors, is added to the angular velocity of the flywheel. In other words, the above-mentioned errors prevent one from guaranteeing that the angular velocity of the flywheel is equal to zero in the equilibrium position. Moreover, the magnitude (6.5) of the angular velocity of the flywheel can be used as a correction to the control in order to compensate for the effect of errors of the telecamera.

7. RESULTS OF CALCULATIONS

We select the following numerical values of the system parameters

$$l = 1\ \text{m}, \quad m_0 = 3\ \text{kg}, \quad c_1 = -0.8\ \text{Nm/V}, \quad J_m = 0.03\ \text{kg m}^2$$

$$R = 0.05\ \text{m}, \quad M = 2\ \text{kg}, \quad c_2 = 0.76\ \text{N m s}, \quad J = 0.297\ \text{kg m}^2$$

$$a = 0.55\ \text{m}, \quad m = 3\ \text{kg}, \quad J_u = 10^{-4}\ \text{kg m}^2, \quad J_0 = 3.75 \times 10^{-3}\ \text{kg m}^2$$

$$\gamma = -10, \quad u_0 = 12\ \text{V}$$

For these parameters we obtain the roots $\mu_1 \approx -7.3s^{-2}$, $\mu_2 \approx -11.5s^{-2}$, and the coefficients in formula (5.14) are found to be: $a_\alpha \approx -3.6$ V, $a_\beta \approx -4.45$ V. As a result, the domain of controllability of the linear system obtained is exceedingly large. For example, if $\beta = \dot{\alpha} = \dot{\beta} = 0$, the value of the angle α can lie completely in the range from $-\pi$ to π, which is considerably wider than the range of its reasonable values of $-\pi/2 < \alpha < \pi/2$.

The criterion (4.6) is satisfied in the case of the chosen values of the system parameters. This means that the eigenvalues (3.9) can be chosen in such a way as to ensure that the coefficient k_4 is equal to zero. With this aim, we select the constants (4.7), for example, to be the following

$$\bar{\lambda}_1 = -15, \quad \bar{\lambda}_2 = -16, \quad \bar{\lambda}_3 = -17, \quad \bar{\lambda}_4 = -18, \quad \bar{\lambda}_5 = -19$$

Using formula (4.8), we find the correction coefficient $\chi \approx 0.834$. Hence, the transient, corresponding to the case when the measurements of the quantity $\dot{\beta}$ can be invoked, is found to be quite fast. The corresponding amplification factors take the values

$$k_1 = -0.816 \times 10^5, \quad k_2 = 0.101 \times 10^5, \quad k_4 = 0$$

$$k_3 = -0.139 \times 10^5, \quad k_5 = -0.337 \times 10^3$$

If one drops the requirement that $k_4 = 0$, the stabilization process can be made more quiescent. In particular, it is possible, for example, to take

$$\lambda_1 = -1, \quad \lambda_2 = -1.01, \quad \lambda_3 = -1.02, \quad \lambda_4 = -1.03, \quad \lambda_5 = -1.04$$

The following amplification factors correspond to these characteristic exponents

$$k_1 = -0.64 \times 10^2, \quad k_2 = 0.449 \times 10^1, \quad k_4 = 0.53$$

$$k_3 = -0.144 \times 10^2, \quad k_5 = -0.95$$

We see that the amplification factor k_4 only increases to a slight extent when the absolute values of the characteristic exponents are reduced. At the same time, the control intensity along the other coordinates decreases very noticeably.

This research was supported financially by the Russian Foundation for Basic Research (01-01-00079, 00-15-96135) and the Federal Special-Purpose "Integration" Programme (T0267, B0053).

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Translated by E.L.S.